OPTIMAL AND UNBIASED FIR ESTIMATES OF CLOCK STATE FOR SPACE AND GROUND APPLICATIONS

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INTRODUCTION

It is commonly accepted [2] that clocks forming time scales in electronic systems have three states: time interval error (TIE), fractional frequency offset, and linear frequency drift rate. All other states are included to the model noise part. If the clock state is estimable, then its behavior can be predicted that is required for many applications. To estimate clock state optimally, several algorithms have been examined for decades. The state space strategy was originally proposed for clocks by Allan and Barnes [2] in order to exploit facilities of Kalman filtering [3]. A description of the clock model was then given by Tryon and Jones in [4] and Chaffee in [5]. Thereafter, the Kalman algorithm has been investigated by many authors [6]—[9] and its applications to timescales were outlined in [10]. The main problem we meet here is associated with clock noise that is inherently colored, whereas Kalman claims the noise to be white sequence. Moreover, the Kalman estimate may diverge [11] in the presence of model uncertainties and high order states and temporary measurement uncertainties caused by the Global Positioning System (GPS), for example.

To overcome problems with the Kalman estimate, transversal finite impulse response (FIR) filtering was originally employed by Shmaliy in [12] in order to design an unbiased estimator required in [1]. In the sequel, the unbiased FIR filters have been investigated in detail and developed in [13]—[15]. It was concluded, both theoretically and experimentally, that the unbiased and optimal FIR solutions converge and become indistinguishable in the GPS-based timekeeping when the averaging interval is composed of a large number N of the points [16] or the clock initial mean square state dominates the noise components in the order of magnitudes [17]. Otherwise, optimal estimators are required. An optimal FIR filter was then proposed in [18] and modified for clocks in [17]. Although [17] opens new horizons in optimal filtering of clock state, the solution proposed requires an initial state that cannot be handled heuristically. Moreover, it does not solve the problems with prediction and smoothing of clock errors. Below, we show and investigate a general full horizon p-shift optimal FIR estimator of clock state at any present ($p = 0$), future ($p > 0$), or past ($p < 0$) time point.

Clock state-space model

The TIE x_n is commonly measured in discrete time n as s_n by a precise time interval counter with a sampling (averaging) time τ for some reference time scale, GPS-based [19], for example. The measurement noise v_n is typically negligible with direct measurement and accounted for if the reference source is wirelessly removed, as in the GPS-based timekeeping. Following [1], a clock can be characterized with three states and its model represented in state space with the state and observation equations, respectively,

$$\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} + \mathbf{w}_n , \tag{1}$$

$$s_n = \mathbf{C}\mathbf{x}_n + v_n , \tag{2}$$

where the 3×1 state vector is given by $\mathbf{x}_n = [x_n \quad y_n \quad z_n]^T$, in which x_n is the TIE, y_n is the fractional frequency offset, and z_n is the linear frequency drift rate. The 3×3 matrix $\mathbf{A}$ projects the nearest past state $\mathbf{x}_{n-1}$ to the present state $\mathbf{x}_n$ employing the finite-degree Taylor series expansions,

$$\mathbf{A} = \begin{bmatrix} 1 & \tau & \tau^2/2 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

and the 1×3 observation noise matrix is $\mathbf{C} = [1 \quad 0 \quad 0]$ for the measurement s_n of x_n. The 3×1 vector of the clock zero-mean noise, $\mathbf{w}_n = [w_{xn} \quad w_{yn} \quad w_{zn}]^T$ has the covariance matrix $\mathbf{\Psi}_n = E\{\mathbf{w}_i \mathbf{w}_j^T\}$ with supposedly arbitrary components for all i and j. It follows from [5] that $\mathbf{\Psi}_n$ can be represented in the white Gaussian approximation via the diffusion coefficients q_1, q_2, and q_3 specified by the white frequency modulation (FM) noise (WHFM), white random walk FM noise (WRFM), and white random run FM noise (RRFM), respectively, in the τ-domain power law of the clock's oscillator. The measurement noise v_n is commonly zero-mean, $E\{v_n\} = 0$, with the variance $E\{v_n^2\} = \sigma_n^2$. It is implied that v_n can have arbitrary distribution and covariance, although it is often delta-correlated.

For the clock model that is suggested in [1] to be distinct over all observation time, the estimation algorithm must be applied to all the measurement data available at once. That means that (1) and (2) need to be modified on an averaging horizon of N points to represent measurement from the first point at zero up to the last one at $n = N-1$. Such a modification has been provided in [17] using recursively computed forward-in-time solutions [20] leading to the time-varying model

$$\mathbf{X}_n = \mathbf{A}_n \mathbf{x}_0 + \mathbf{B}_n \mathbf{N}_n, \tag{4}$$

$$\mathbf{S}_n = \mathbf{C}_n \mathbf{x}_0 + \mathbf{G}_n \mathbf{N}_n + \mathbf{V}_n, \tag{5}$$

where $\mathbf{X}_n = [\mathbf{x}_n^T \quad \mathbf{x}_{n-1}^T \quad \cdots \quad \mathbf{x}_0^T]^T$, $\mathbf{S}_n = [s_n \quad s_{n-1} \quad \cdots \quad s_0]^T$, $\mathbf{N}_n = [\mathbf{w}_n^T \quad \mathbf{w}_{n-1}^T \quad \cdots \quad \mathbf{w}_0^T]^T$, $\mathbf{V}_n = [v_n \quad v_{n-1} \quad \cdots \quad v_0]^T$, $(\mathbf{A}^i)_1 = [1 \quad \tau i \quad \tau^2 i^2/2]$,

$$\mathbf{A}_n = [\mathbf{A}^{n^T} \quad \mathbf{A}^{n-1^T} \quad \cdots \quad \mathbf{A}^T \quad \mathbf{I}]^T, \tag{6}$$

$$\mathbf{B}_n = \begin{bmatrix} \mathbf{I} & \mathbf{A} & \cdots & \mathbf{A}^{n-1} & \mathbf{A}^n \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{A}^{n-2} & \mathbf{A}^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{7}$$

$$\mathbf{C}_n = [(\mathbf{A}^n)_1^T \quad (\mathbf{A}^{n-1})_1^T \quad \cdots \quad (\mathbf{A})_1^T \quad (\mathbf{I})_1^T]^T, \tag{8}$$

$$\mathbf{G}_n = \begin{bmatrix} (\mathbf{I})_1 & (\mathbf{A})_1 & \cdots & (\mathbf{A}^{n-1})_1 & (\mathbf{A}^n)_1 \\ \mathbf{0} & (\mathbf{I})_1 & \cdots & (\mathbf{A}^{n-2})_1 & (\mathbf{A}^{n-1})_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & (\mathbf{I})_1 & (\mathbf{A})_1 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{I})_1 \end{bmatrix}, \tag{9}$$

where $(\mathbf{Z})_1$ means the first row of a matrix $\mathbf{Z}$. Here $\mathbf{I}$ is identity and $\mathbf{0}$ is a relevant matrix with all components equal to zero. Note that the initial state $\mathbf{x}_0$ is supposed to be given exactly, although it is randomly-valued. Therefore, $\mathbf{w}_0$ in this model is zero-valued.

Full horizon *p*-shift optimal FIR estimator

An optimal FIR estimate $\tilde{\mathbf{x}}_{n+p|n}$ of the *K*-state clock vector $\mathbf{x}_{n+p}$ can be provided at $n+p$ if we assign some $K \times N$ gain matrix $\mathbf{H}(p)$, apply it to $\mathbf{S}_n$, employ (5), and claim that

$$\tilde{\mathbf{x}}_{n+p|n} = \mathbf{H}(p)\mathbf{S}_n = \mathbf{H}(p)(\mathbf{C}_n\mathbf{x}_0 + \mathbf{G}_n\mathbf{N}_n + \mathbf{V}_n)\,. \tag{10}$$

Because $\mathbf{S}_n$ in (10) can be considered as an input and $\tilde{\mathbf{x}}_{n+p|n}$ an output of the estimator, the gain matrix $\mathbf{H}(p)$ would realize the convolution principle. For $\mathbf{H}(p)$ to be optimal in the minimum mean square error (MSE) sense, the components of the error matrix $\mathbf{J}(p) = E\{(\mathbf{x}_{n+p} - \tilde{\mathbf{x}}_{n+p|n})(\mathbf{x}_{n+p} - \tilde{\mathbf{x}}_{n+p|n})^T\}$ must be minimized. It has been shown in [17,18] that a minimization can be achieved using the orthogonality condition [20] in the form of

$$\mathbf{0} = E\left\{[x_{n+p} - \tilde{\mathbf{H}}(p)(\mathbf{C}_n\mathbf{x}_0 + \mathbf{G}_n\mathbf{N}_n + \mathbf{V}_n)](\mathbf{C}_n\mathbf{x}_0 + \mathbf{G}_n\mathbf{N}_n + \mathbf{V}_n)^T\right\} \tag{11}$$

to produce the optimal gain matrix $\tilde{\mathbf{H}}(p)$. In order to find this gain, the clock state vector $\mathbf{x}_{n+p}$ needs to be substituted with the deterministic model $\mathbf{x}_{n+p} = \mathbf{A}^{n+p}\mathbf{x}_0$. Supposing that the initial state and the measurement noise are mutually uncorrelated and independent processes for all p, one can provide averaging in (9) and arrive at the optimal gain matrix

$$\tilde{\mathbf{H}}(p) = \mathbf{A}^{n+p}\mathbf{R}_0\mathbf{C}_n^T(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}, \tag{12}$$

in which auxiliary matrices are $\mathbf{Z}_0 = \mathbf{C}_n\mathbf{R}_0\mathbf{C}_n^T$ and $\tilde{\mathbf{Z}}_\Psi = \mathbf{G}_n\mathbf{\Psi}_N\mathbf{G}_n^T$, the initial mean square state is specified with $\mathbf{R}_0 = E\{\mathbf{x}_0\mathbf{x}_0^T\}$, and the signal and measurement noise covariance function matrices are given by, respectively, $\mathbf{\Psi}_N = E\{\mathbf{N}_n\mathbf{N}_n^T\}$ and $\mathbf{\Phi}_V = E\{\mathbf{V}_n\mathbf{V}_n^T\}$. Further multiplying $\mathbf{R}_0$ in (10) from the left-hand side with the identity matrix $(\mathbf{C}_n^T\mathbf{C}_n)^{-1}\mathbf{C}_n^T\mathbf{C}_n$ brings the optimal gain to its most compact and general form of

$$\tilde{\mathbf{H}}(p) = \bar{\mathbf{H}}(p)\mathbf{Z}_0(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}, \tag{13}$$

in which

$$\overline{\mathbf{H}}(p) = \mathbf{A}^{n+p}(\mathbf{C}_n^T\mathbf{C}_n)^{-1}\mathbf{C}_n^T \tag{14}$$

is the unbiased gain associated with the noiseless clock and measurement. Note that (12) is computable for $n \geq K$, because the inverse in (12) does not exist otherwise. It can easily be verified that (12) fits the unbiasedness condition $E\{\tilde{\mathbf{x}}_{n+p|n}\} = E\{\mathbf{x}_{n+p}\}$. Indeed, employ (12) in (8) and average both sides of (8). Averaging means removing the covariance matrices of the zero mean noise components and we arrive at unbiasedness condition. In applications to clock problems, the unbiased gain has been investigated in [12]—[15]. Its splendid property is in the independence on noise and initial conditions. Moreover, it becomes practically optimal when $N >> 1$ (this case is typical for GPS-based timekeeping) and when the initial mean square clock state matrix $\mathbf{Z}_0$ dominates the noise component matrices, $\tilde{\mathbf{Z}}_\Psi$ and $\mathbf{\Phi}_V$, in the order of magnitudes [17] (this case can be met in unlocked clocks).

In order to find an optimal estimate using (8) with (11), the initial mean square clock state $\mathbf{R}_0$ needs to be ascertained. That can be provided by optimal smoothing the measurement data at zero. Namely, letting $p = -n$ in (11) and employing (8) allows us to find an optimal smooth at $n = 0$ as

$$\tilde{\mathbf{x}}_{0|n} = \mathbf{R}_0\mathbf{C}_n^T(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}\mathbf{S}_n \tag{15}$$

For $\tilde{\mathbf{x}}_{0|n}$ estimated optimally, we can let $\mathbf{x}_0 = \tilde{\mathbf{x}}_{0|n}$. Since the initial state $\mathbf{x}_0$ is supposed to be given exactly, we can also accept $\mathbf{R}_0 = E\{\mathbf{x}_0\mathbf{x}_0^T\} = \mathbf{x}_0\mathbf{x}_0^T \cong E\{\tilde{\mathbf{x}}_{0|n}\tilde{\mathbf{x}}_{0|n}^T\} \cong \tilde{\mathbf{x}}_{0|n}\tilde{\mathbf{x}}_{0|n}^T$. That leads to

$$\mathbf{R}_0 \cong \mathbf{R}_0\mathbf{C}_n^T(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}\mathbf{S}_n\mathbf{S}_n^T(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}\mathbf{C}_n\mathbf{R}_0 \tag{16}$$

and, if we accept an equality in (14), to the discrete algebraic Riccati equation (DARE)

$$\mathbf{0} = \mathbf{Z}_0(\tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}\mathbf{Z}_0 + 2\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V - \mathbf{S}_n\mathbf{S}_n^T(\tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}\mathbf{Z}_0 \,, \tag{17}$$

which solution with respect to $\mathbf{Z}_0$ can be found following [21].

The optimal FIR estimate $\tilde{\mathbf{x}}_{n+p|n}$ of clock state can now be generalized at $n + p$ as in the following. Given the clock model, (1) and (2), and determined $\mathbf{Z}_0$, by solving the DARE (14), then filtering $(p = 0)$, p-lag smoothing $(p < 0)$, and p-step prediction $(p > 0)$ of clock state can be provided at $n + p$ in the minimum MSE sense employing measurement s_n of the TIE taken from zero to n by

$$\tilde{\mathbf{x}}_{n+p|n} = \tilde{\mathbf{H}}(p)\mathbf{S}_n = \overline{\mathbf{H}}(p)\mathbf{Z}_0(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}\mathbf{S}_n \tag{18}$$

$$= \mathbf{A}^{n+p}(\mathbf{C}_n^T\mathbf{C}_n)^{-1}\mathbf{C}_n^T\mathbf{Z}_0(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi + \mathbf{\Phi}_V)^{-1}\mathbf{S}_n \,. \tag{19}$$

One can observe that, contrary to the Kalman filter, the optimal FIR estimate (19) does not require the initial state and self-determines the initial mean square state, by solving the DARE. Moreover, it allows each of the noise components to have arbitrary distribution and covariance functions. Thus, it can directly be applied to clocks and measurements having nonwhite noise sources. In a specific case of $N >> 1$, (19) can be substituted with the unbiased estimate

$$\bar{\mathbf{x}}_{n+p|n} = \mathbf{A}^{n+p}(\mathbf{C}_n^T\mathbf{C}_n)^{-1}\mathbf{C}_n^T\mathbf{S}_n \tag{20}$$

to produce still a nice near optimal estimate.

Applications

Below, we give several typical applications to clocks. We provide estimation of the current state of the United States Naval Observatory (USNO) and National Institute of Standards and Technology (NIST) Master Clocks (MCs) employing data published on their WEB sites.

Estimation of the USNO MC

The USNO has published on the WEB site the UTC–UTC(USNO MC) time differences (73 points) measured each 5 days in 2008, as issued monthly by BIPM [units are in Modified Julian Dates (MJDs) and nanoseconds]. For this measurement, we form the time scale starting with $n = 0$ (54464.0 MJD) and finishing at $n = 72$ (54829.0 MJD). The measurement is shown in Fig. 1a (circles). Because the frequency drift in the USNO MC is negligibly small and measurement is provided with negligible errors, the clock can be represented with the two-state space model, $\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} + \mathbf{w}_n$ and $s_n = \mathbf{C}\mathbf{x}_n$, in which $\mathbf{x}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Since the USNO MC time scale is corrected, the time error noise can be considered to be zero-mean and white. We calculate the variance of this noise as $\sigma_{xn}^2 = E\{w_n^2\}$ providing the averaging from zero to the current point n. The second clock state can be represented by the time derivative of the first state and the noise variance calculated similarly. Accordingly, we form the clock noise vector as $\mathbf{w}_n = \begin{bmatrix} w_{xn} & w_{yn} \end{bmatrix}^T$ and the covariance matrix with $\mathbf{\Psi}_n = \begin{bmatrix} \sigma_{xn}^2 & E\{w_{xn}w_{ym}\} \\ E\{w_{yn}w_{xm}\} & \sigma_{yn}^2 \end{bmatrix}$, where $0 \le m \le n$. Note that the effect of frequency noise is relatively small in $\mathbf{\Psi}_n$. For this clock, the optimal FIR filtering estimate can be provided, with $p = 0$, by modifying (17) and (19) to, respectively,

$$\mathbf{0} = \mathbf{Z}_0\tilde{\mathbf{Z}}_\Psi^{-1}\mathbf{Z}_0 + 2\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi - \mathbf{S}_n\mathbf{S}_n^T\tilde{\mathbf{Z}}_\Psi^{-1}\mathbf{Z}_0\ , \tag{21}$$

$$\hat{\mathbf{x}}_{n|n} = \mathbf{A}^n(\mathbf{C}_n^T\mathbf{C}_n)^{-1}\mathbf{C}_n^T\mathbf{Z}_0(\mathbf{Z}_0 + \tilde{\mathbf{Z}}_\Psi)^{-1}\mathbf{S}_n\,. \tag{22}$$

The unbiased estimate (20) accordingly becomes

$$\bar{\mathbf{x}}_{n|n} = \mathbf{A}^n(\mathbf{C}_n^T\mathbf{C}_n)^{-1}\mathbf{C}_n^T\mathbf{S}_n\,. \tag{23}$$

Figure 1 (a)—(c) sketches the estimates of the USNO MC x_n and y_n provided by the optimal estimator, (21) and (22), and unbiased one (23). In turn, Fig. 1d gives us the relevant differences in % between the optimal and unbiased estimates of the first state. We notice that similar errors can be observed as related to the second state. Estimates of the first state are illustrated in Fig. 1a and Fig. 1d. These figures reveal that $\hat{\mathbf{x}}_{n|n}$ and $\bar{\mathbf{x}}_{n|n}$ differ in average on about 15 % and that this measure can reach 50 % at some points. Although a comparison of optimal and unbiased estimates is a special topic, there is an immediate explanation to differences with small N. The unbiased filter provides the best fit for the noisy process, whereas in the optimal filter this fit is adjusted by the noise covariance function. On the one hand, the latter cannot be ascertained correctly with a small number of measurements. Therefore, the optimal filter may produce errors. On the other hand, the unbiased filter is associated with large horizons and may not be precise otherwise. So, we have an inconsistency that would be reduces by increasing N.

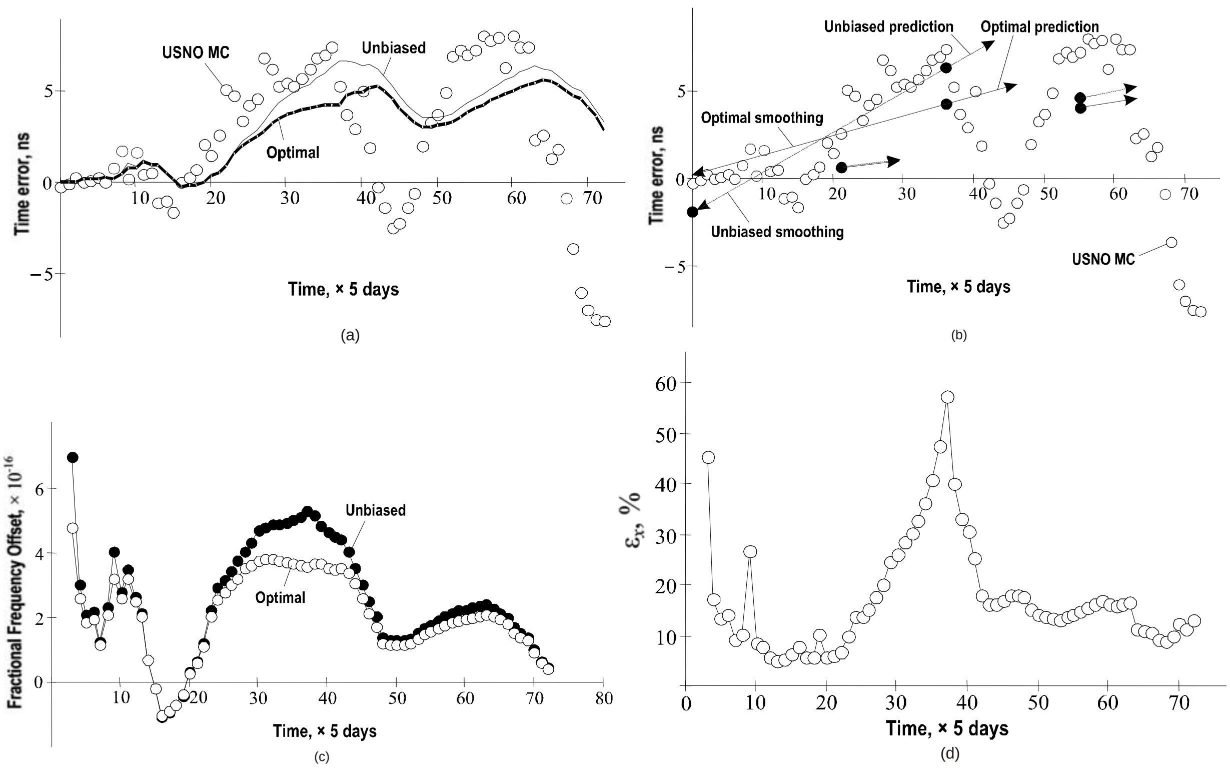

Fig. 1. Optimal and unbiased estimates of the USNO MC via measurement in 2008: (a) filtering of the first state, (b) prediction and smoothing of the first state, (c) filtering of the second state, and (d) difference in % between the optimal and unbiased filtering estimates of the first state.

Prediction and smoothing of time errors is illustrated in Fig. 1b. Here, we employ the p-shift general optimal algorithm (19) with $\mathbf{Z}_0$ determined by solving the DARE (17) and the p-shift unbiased algorithm (20). For the illustrative purposes, we fixed three time points, n = 21, n = 36, and n = 56, and allowed negative p to provide smoothing and positive p to obtain prediction. Prediction and smoothing are depicted in Fig. 1b with the right and left arrows, respectively. It can be shown that the unbiased prediction is exactly that found in [22] employing a ramp FIR filter. In [22], one can also find an unbiased prediction of clock errors obtained with a 5-point step and unbiased smoothing of measurement providing the best fit. An important observation can be made observing Fig. 1b. It is seen that the unbiased smoothing function ranging from n = 36 to zero fits better than the optimal one. However, this fact does not mean that the optimal smooth is lesser accurate, because the latter fits better the full measurement. Finally, Fig. 1c sketches filtering estimates of the second clock state provided with the optimal and unbiased filters. Again we infer that both estimates are consistent, except for the region from n = 25 to n = 45, in which the discrepancy reaches 50 %.

Estimation of the NIST MC

The NIST has published on the WEB site the UTC—UTC(NIST MC) time differences (285 points) measured each 10 days in 2002—2009, as issued monthly by BIPM [units are in Modified Julian Dates (MJDs) and nanoseconds]. For this measurement, we form the time scale starting with $n = 0$ (52279 MJD) and finishing at $n = 284$ (55129 MJD). The measurement is shown in Fig. 2a (circles).

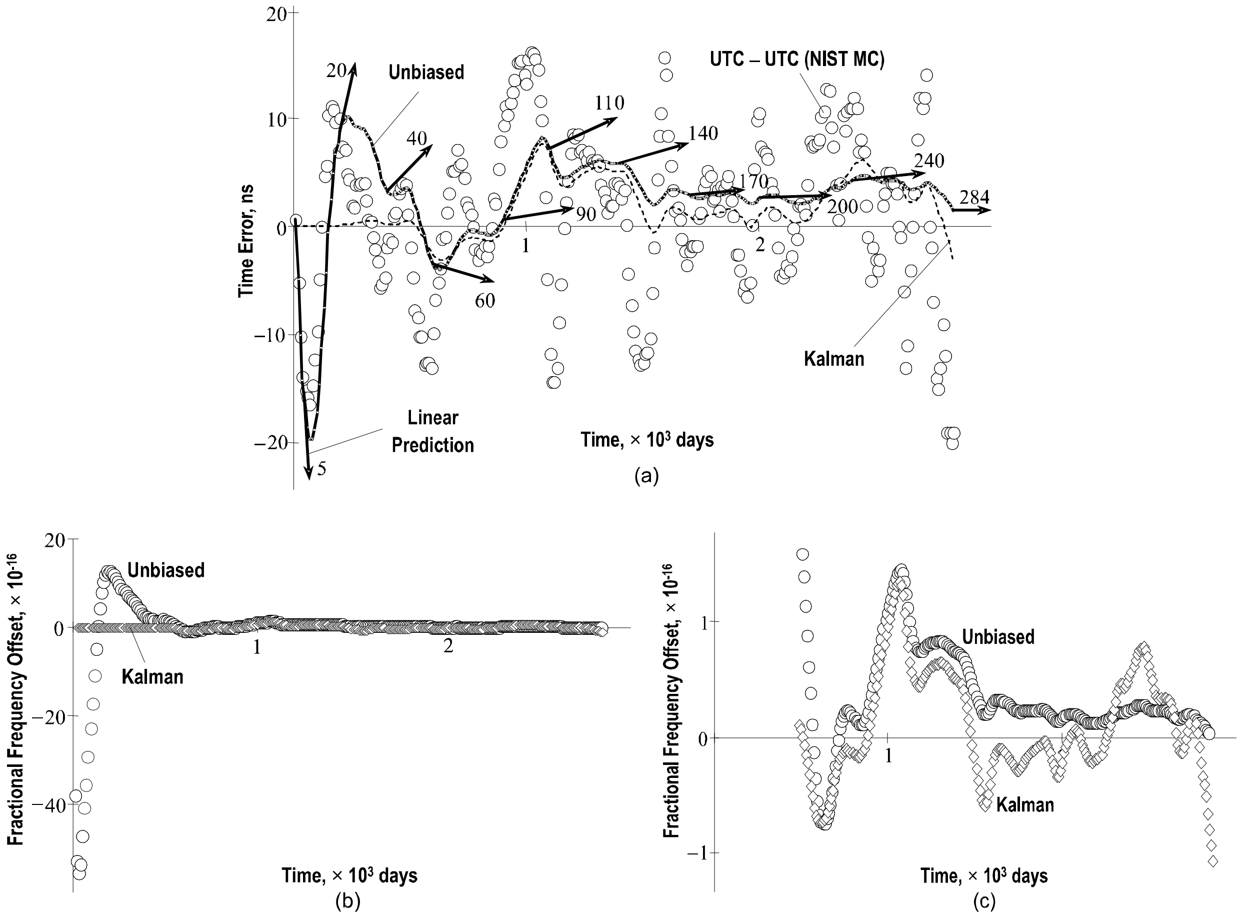

Fig. 2. Unbiased FIR and Kalman estimates of the NIST MC via measurement in 2002—2009: (a) filtering and prediction of the first state, (b) filtering of the second state, (c) filtering of the second state without transients.

Supposing that the NIST MC has two states, $K = 2$, the unbiased algorithm (23) was applied simultaneously with the 2-state Kalman filter described in [12] (see Appendix B). Figure 2 sketches the filtering estimates of the first and second states. For a comparison, we also give estimates obtained with the standard 2-state Kalman filter. Because the NIST MC exhibits excellent etalon properties, we set zeroth initial conditions, $x_0 = 0$ and $y_0 = 0$. For the resolution of 0.1 ns in the published data, the digitization noise was supposed to be distributed uniformly and its variance calculated as $\sigma_v^2 = 0.05^2/3$ ns^2 $= 8.33\times10^{-4}$ ns^2. Current estimates of the first state are given in Fig. 2a along with the linear prediction of time errors depicted with arrows. Since the unbiased estimator gives us the best linear fit, the first arrow coincides in the direction with several initial measurement points. Increased n, prediction depicts possible behaviors of future errors. Although, the process shown in Fig. 2a looks like rather stationary, prediction reveals some positive average angle resulting in a positive frequency offset diminishing with time (Fig. 2b and Fig. 2c).

An analysis shows that the Kalman filter is lesser suitable for the clock current state estimation. In fact, owing to transients, the Kalman filter does not provide a real picture with a small number of measurements. Although both filters produce consistent estimates in the intermediate region (about the point of 1×10^3 days), the unbiased algorithm keeps smoothing the data beyond this region. In turn, the Kalman filter exhibits qualitatively the same picture, being not dependent on the observation interval.

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